PART – III

MATHEMATICS (MARCH 2015)

TIME: 3 Hrs **Max. Marks: 200**

PART – A

Note: i) **All** Questions are **Compulsory**.
ii) Choose the most suitable answer from the given four alternatives and write the option code and the corresponding answer.

40 x 1 = 40

1. If the equation $-2x + y + z = l, \quad x - 2y + z = m, \quad x + y - 2z = n$ such that $l + m + n = 0$ then the system has:
   a. a non-zero unique solution
   b. trivial solution
   c. infinitely many solution
   d. no solution
   Ans. (c)
2. If x is a continuous random variable then $P(a < X < b) =$
   a. $P(a \le X \le b)$
   b. $P(a < X \le b)$
   c. $P(a \le X < b)$
   d. all the three above
   Ans. (d)
3. $p + iq = (2 - 3i) \quad (4 + 2i)$ then q is:
   a. 14
   b. -14
   c. -8
   d. 8
   Ans. (c)
4. The Point of intersection of the lines $\vec{r} = \left(-\vec{i} + 2\vec{j} + 3\vec{k}\right) + t\left(-2\vec{i} + \vec{j} + \vec{k}\right)$ *and* $\vec{r} = \left(2\vec{i} + 3\vec{j} + 5\vec{k}\right) + s\left(\vec{i} + 2\vec{j} + 3\vec{k}\right)$ *is :*
   a. (2, 1, 1)
   b. (1, 2, 1)
   c. (1, 1, 2)
   d. (1, 1, 1)
   Ans. (c)
5. $p \leftrightarrow q$ is equivalent to:
   a. $p \to q$
   b. $q \to q$
   c. $(p \to q) \vee (q \to p)$
   d. $(p \to q) \wedge (q \to p)$
   Ans. (d)

6. If 2 cards are drawn from a well shuffled pack of 52 cards, the probability that they are of the same colors without replacement is:
   a. $\frac{1}{2}$
   b. $\frac{26}{51}$
   c. $\frac{25}{51}$
   d. $\frac{25}{102}$ Ans. (c)
7. The Curve $9y^2 = x^2(4 - x^2)$ is symmetrical about:
   a. $y - axis$
   b. $x - axis$
   c. $y = x$
   d. both the axes Ans. (d)
8. The point of inflexion of the curve $y = x^4$ is at:
   a. $x = 0$
   b. $x = 3$
   c. $x = 12$
   d. Nowhere Ans. (d)
9. Given $E(X + c) = 8 \; and \; E(X - c) = 12$ then the value of c is:
   a. – 2
   b. 4
   c. – 4
   d. 2 Ans. (a)
10. Volume of solid obtained by revolving the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about major and minor axes are in the ratio:
   a. $b^2 : a^2$
   b. $a^2 : b^2$
   c. $a : b$
   d. $b : a$ Ans. (d)
11. If a compound statement is made up of three simple statements, then the number of rows in the truth tale is:
   a. 8
   b. 6
   c. 4
   d. 2 Ans. (a)

12. In the group $(Z_5 - \{[0]\}, \bullet 5)$, $o([3])$ is:

a. 5
b. 3
c. 4
d. 2 — Ans. (c)

13. The length of the arc of the curve $x^{2/3} + y^{2/3} = 4$ is:

a. 48
b. 24
c. 12
d. 96 — Ans. (a)

14. If $x = \cos\theta + i\sin\theta$ the value of $x^n + \frac{1}{x^n}$ is:

a. $2\cos n\theta$
b. $2i\sin n\theta$
c. $2\sin n\theta$
d. $2i\cos n\theta$ — Ans. (a)

15. The value of $[\vec{i} + \vec{j},\ \vec{j} + \vec{k},\ \vec{k} + \vec{i}]$ is equal to:

a. 0
b. 1
c. 2
d. 4 — Ans. (c)

16. If $\vec{u} = \vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b})$, then:

a. $\vec{u}$ is a unit vector
b. $\vec{u} = \vec{a} + \vec{b} + \vec{c}$
c. $\vec{u} = \vec{0}$
d. $\vec{u} \neq \vec{0}$ — Ans. (c)

17. If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$ then:

a. $\vec{a}$ is parallel to $\vec{b}$
b. $\vec{a}$ is perpendicular to $\vec{b}$
c. $|\vec{a}| = |\vec{b}|$
d. $\vec{a}$ and $\vec{b}$ are unit vectors — Ans. (b)

18. If $\rho(A) = \rho[A, B]$ then the system is:

a. Consistent and has infinitely many solution.
b. Consistent and has a unique solution
c. Consistent
d. Inconsistent — Ans. (c)

19. The Complementary function of $(D^2+1)y=e^{2x}$ is:

a. $(Ax+B)e^{x}$
b. $A\cos x+B\sin x$
c. $(Ax+B)e^{2x}$
d. $(Ax+B)e^{-x}$ Ans. (b)

20. If f(x) is an odd function, then $\int_{-a}^{a} f(x)dx$ is:

a. $2\int_{0}^{a} f(x)dx$
b. $\int_{0}^{a} f(x)dx$
c. 0
d. $\int_{-a}^{a} f(a-x)dx$ Ans. (c)

21. The length of the semi-major and the length of the semi-minor axis of the ellipse $\frac{x^2}{144}+\frac{y^2}{169}=1$ are:

a. 26, 12
b. 13, 24
c. 12, 26
d. 13, 13 Ans. (d)

22. The differential equation of all circles with centre at the origin is:

a. $x\,dy+y\,dx=0$
b. $x\,dy-y\,dx=0$
c. $x\,dx+y\,dy=0$
d. $x\,dx-y\,dy=0$ Ans. (c)

23. The function $f(x)=x^2$ is decreasing in:

a. $(-\infty,\infty)$
b. $(-\infty,0)$
c. $(0,\infty)$
d. $(-2,\infty)$ Ans. (b)

24. For a Poisson distribution with parameter $\lambda=0.25$ the value of the 2nd moment about the origin is:

a. 0.25
b. 0.3125
c. 0.0625
d. 0.025 Ans. (b)

25. The solution of a linear differential equation $\frac{dx}{dy} + Px = Q$ where P and Q are function of y, is:

a. $y(I.F) = \int (I.F)Q\, dx + c$
b. $x(I.F) = \int (I.F)Q\, dy + c$
c. $y(I.F) = \int (I.F)Q\, dy + c$
d. $x(I.F) = \int (I.F)Q\, dx + c$ Ans. (b)

26. The curve $ay^2 = x^2(3a - x)$ cuts the y-axis at:

a. $x = -3a, x = 0$
b. $x = 0, x = 3a$
c. $x = 0, x = a$
d. $x = 0$ Ans. (d)

27. If A and B are any two matrices such that AB = 0 and A is non-singular, then:

a. B = 0
b. B is singular
c. B is non-singular
d. B = A Ans. (a)

28. The eccentricity of the hyperbola whose latus rectum is equal to half of its conjugate axis is:

a. $\frac{\sqrt{3}}{2}$
b. $\frac{5}{3}$
c. $\frac{3}{2}$
d. $\frac{\sqrt{5}}{2}$ Ans. (d)

29. The Value of $\int_0^1 x(1 - x)^4 dx$ is:

a. $\frac{1}{12}$
b. $\frac{1}{30}$
c. $\frac{1}{24}$
d. $\frac{1}{20}$ Ans. (b)

30. In the law of mean, the value of 'θ' satisfies the condition:

a. $\theta > 0$
b. $\theta < 0$

c. $\theta > 1$

d. $0 < \theta < 1$ Ans. (d)

31. The modulus and amplitude of the complex number $\left[e^{3-i\pi/4}\right]$ are respectively:

a. $e^9, \frac{\pi}{2}$

b. $e^9, \frac{-\pi}{2}$

c. $e^6, \frac{-3\pi}{4}$

d. $e^9, \frac{-3\pi}{4}$ Ans. (d)

32. Which of the following is not a binary operation on R?

a. $a * b = ab$

b. $a * b = a - b$

c. $a * b = \sqrt{ab}$

d. $a * b = \sqrt{a^2 + b^2}$ Ans. (c)

33. The slope of the normal to the parable $y = 3x^2$ at the point whose x-coordinate 2 is:

a. $\frac{1}{13}$

b. $\frac{1}{14}$

c. $\frac{-1}{12}$

d. $\frac{1}{12}$ Ans. (c)

34. The line $4x + 2y = c$ is a tangent to the parabola $y^2 = 16x$, then c is:

a. -1

b. -2

c. 4

d. -4 Ans. (b)

35. If the matrix $\begin{bmatrix} -1 & 3 & 2 \\ 1 & k & -3 \\ 1 & 4 & 5 \end{bmatrix}$ has an inverse then the values of k:

a. k is any real number

b. $k = -4$

c. $k \neq -4$

d. $k \neq 4$ Ans. (c)

36. If θ is the angle between the line $\vec{r} = \vec{a} + t\vec{b}$ and the plane $\vec{r} \bullet \vec{n} = q$ is connected by the relation:

a. $\cos\theta = \dfrac{\vec{a} \bullet \vec{n}}{q}$

b. $\cos\theta = \dfrac{\vec{b} \bullet \vec{n}}{|\vec{b}||\vec{n}|}$

c. $\sin\theta = \dfrac{\vec{a} \bullet \vec{b}}{|\vec{n}|}$

d. $\sin\theta = \dfrac{\vec{a} \bullet \vec{b}}{|\vec{b}||\vec{n}|}$ Ans. (c)

37. The degree of the differential equation $\sqrt{1 + \left(\dfrac{dy}{dx}\right)^{1/3}} = \dfrac{d^2y}{dx^2}$:

a. 1
b. 2
c. 3
d. 6 Ans. (d)

38. If w is the n^{th} root of unity then:

a. $1 + w^2 + w^4 + \cdots = w + w^2 + w^5 + \cdots$
b. $w^n = 0$
c. $w^n = 1$
d. $w = w^{n-1}$ Ans. (c)

39. The work done by the force $\vec{F} = a\vec{i} + \vec{j} + \vec{k}$ in moving the point of application from (1, 1, 1) to (2, 2, 2) along a straight line is given to be 5 units. The value of a is:

a. – 3
b. 3
c. 8
d. – 8 Ans.(b)

40. The condtition that the line $lx + my + nz = 0$ may be a tangent to the rectangular hyperbola $xy = c^2$ is:

a. $a^2l^2 + b^2m^2 = n^2$
b. $am^2 = ln$
c. $a^2l^2 - b^2m^2 = n^2$
d. $4c^2lm = n^2$ Ans. (d)

PART – B

Note:

i) Answer **any ten** questions

ii) Question No.**55** is **Compulsory** and choose **any nine** question from the remaining.

10 x 6 = 60

41. Find the rank of the matrix $\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$

Solution:

$$Let A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1 & 2 & 4 & 3 \\ 4 & 3 & 6 & 7 \\ 0 & 1 & 2 & 1 \end{bmatrix} \quad C_1 \leftrightarrow C_3$$

$$\approx \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & -5 & -10 & -5 \\ 0 & 1 & 2 & 1 \end{bmatrix} \quad R_2 \to R_2 - 4R_1$$

$$\approx \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & -1 & -2 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix} \quad R_2 \to \frac{R_2}{5}$$

$$\approx \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

The last equivalent matrix is in the echelon form. The number of non zero rows = 2.
$\rho(2) = 2.$

42. Solve by matrix inversion method $2x - y = 7 \quad and \quad 3x - 2y = 1$.

Solution:

The given system of equations can be written in the form of

$$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}$$

$$A\ X = B$$

$$|A| = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} = -4 + 3 = -1 \neq 0$$

$$A^{-1} = \frac{1}{|A|} Adj\ A = \frac{1}{-1}\begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$$

$$X = A^{-1}B = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 2(7) + (-1)(11) \\ 3(7) + (-2)(11) \end{bmatrix} = \begin{bmatrix} 14 - 11 \\ 21 - 22 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad \therefore \underline{x = 3};\ \underline{y = -1}$$

43. Find $(\vec{a} \times \vec{b}) \bullet (\vec{c} \times \vec{d})$, *if* $\vec{a} = \vec{i} + \vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} + \vec{k}$, $\vec{c} = 2\vec{i} + \vec{j} + \vec{k}$ *and* $\vec{d} = \vec{i} + \vec{j} + 2\vec{k}$

Solution:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = \vec{i}(1-0) - \vec{j}(1-2) + \vec{k}(0-2) = \vec{i} + \vec{j} - 2\vec{k}$$

$$\vec{c} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \vec{i}(2-1) - \vec{j}(4-1) + \vec{k}(2-1) = \vec{i} - 3\vec{j} + \vec{k}$$

$$(\vec{a} \times \vec{b}) \bullet (\vec{c} \times \vec{d}) = (\vec{i} + \vec{j} - 2\vec{k}) \bullet (\vec{i} - 3\vec{j} + \vec{k}) = 1 - 3 - 2 = -4$$

44 a) If $A(-1, 4, 3)$ **is one end of the diameter AB of the sphere** $x^2 + y^2 + z^2 - 3x - 2y + 2z - 15 = 0$, **then find the co-ordinates of B.**

b) Find the angle between the line $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-3}{-2}$ **and the plane** $3x + 4y + z + 5 = 0$.

Solution:

(a) The equation of sphere

$$x^2 + y^2 + z^2 - 3x - 2y + 2z - 15 = 0$$

$$2u = -3; \quad 2v = -2; \quad 2w = 2; \quad d = -15$$

$$u = -3/2; \quad v = -2/2; \quad w = 1$$

$$Centre = (-u, -v, -w) = (3/2, 1, -1)$$

Given $A(-1, 4, -3)$ *let other end be* $B(x, y, z)$

Mid point of diameter $AB = (3/2, 1, -1)$

$$\left(\frac{x-1}{2},\frac{y+4}{2},\frac{z-3}{2}\right)=\left(\frac{3}{2},1,-1\right)$$

$$\frac{x-1}{2}=\frac{3}{2} \qquad \frac{y+4}{2}=1 \qquad \frac{z-3}{2}=-1$$

$$x-1=3 \qquad y+4=2 \qquad z-3=-2$$

$$x=4 \qquad y=-2 \qquad z=1$$

The other end $B(4,-2,1)$

b) $\vec{b}=3\vec{i}+\vec{j}+\vec{k} \quad \vec{n}=3\vec{i}+4\vec{j}+\vec{k}$

$$\vec{b}\bullet\vec{n}=\left(3\vec{i}+\vec{j}+\vec{k}\right)\bullet\left(3\vec{i}+4\vec{j}+\vec{k}\right)=9-4-2=3$$ -10y

$$\left|\vec{b}\right|=\sqrt{9+1+4}=\sqrt{14}; \quad \left|\vec{n}\right|=\sqrt{9+16+1}=\sqrt{26}$$

Let θ be the angle between line and plane

$$\sin\theta=\frac{\vec{b}\bullet\vec{n}}{\left|\vec{b}\right|\left|\vec{n}\right|}=\frac{3}{\sqrt{14}\sqrt{26}}=\frac{3}{\sqrt{364}}=\frac{3}{2\sqrt{91}}$$

$$\theta=\sin^{-1}\left(\frac{3}{2\sqrt{91}}\right)$$

45. P represents the variable complex number z. Find the locus of P, if $|z-5i|=|z+5i|$

Solution:

Let P represents the variable complex number z. Let $z=x+iy$

Given $|z-5i|=|z+5i|$

$$|x+iy-5i|=|x+iy+5i|$$

$$|x+i(y-5)|=|x+i(y+5)|$$

$$\sqrt{x^2+(y-5)^2}=\sqrt{x^2+(y+5)^2}$$

$$x^2+(y-5)^2=x^2+(y+5)^2$$

$$y^2+25-10y=y^2+25+10y$$

$$-10y-10y=0 \quad \Rightarrow \quad -20y=0 \quad \Rightarrow \quad y=0$$

46. Find the square root of $(-7+24i)$

Solution:

The Square root of $(-7+24i)$

$\sqrt{-7+24i} = (a+ib)$

$(-7+24i) = (a+ib)^2$

$(-7+24i) = a^2 - b^2 + 2iab$

$a^2 - b^2 = -7; \quad 2ab = 24$

$a^2 + b^2 = \sqrt{(a^2-b^2)^2 + 4a^2b^2} \quad \left(\because (a^2-b^2)^2 = a^4 + b^4 - 2a^2b^2\right)$

$\boldsymbol{a^2 + b^2 = \sqrt{(a^2-b^2)^2 + (2ab)^2} = \sqrt{(-7)^2 + (24)^2} = \sqrt{49+576} = \sqrt{625} = 25}$

$\boldsymbol{a^2 + b^2 = 25} \quad (+)$

$\boldsymbol{a^2 - b^2 = -7}$

$\boldsymbol{2a^2 = 18}$

$\boldsymbol{a^2 = 9; a = \pm 3}$

$\boldsymbol{a^2 + b^2 = 25} \quad (-)$

$\boldsymbol{a^2 - b^2 = -7}$

$\boldsymbol{2b^2 = 32}$

$\boldsymbol{b^2 = 16; b = \pm 4}$

$\boldsymbol{\therefore \sqrt{-7+24i} = 3 \pm i4}$

47. a) Evaluate $\lim\limits_{x\to\infty} \frac{x^2}{e^x}$

b) Determine the domain of concavity (Convexity) of the curve $y = 2 - x^2$

Solution:

a) $\lim\limits_{x\to\infty} \frac{x^2}{e^x} \quad \left(form \ \frac{\infty}{\infty}\right)$

Applying L' Hospital rules

$\lim\limits_{x\to\infty} \frac{x^2}{e^x} = \lim\limits_{x\to\infty} \frac{2x}{e^x} \quad \left(form \ \frac{\infty}{\infty}\right)$

Applying L' Hospital rules

$\lim\limits_{x\to\infty} \frac{2}{e^x} = \frac{2}{\infty} = 0$

b) $y = 2 - x^2$

$\frac{dy}{dx} = 0 - 2x = -2x$

$\frac{d^2y}{dx^2} = -2 < 0 \qquad \forall\, x \in R$

The curve is concave downward for $x \in R$

48. At what angle θ do the curve $y = a^x$ *and* $y = b^x$ intersect $(a \neq b)$?

Solution:

The given curves are $y = a^x \to (1); \quad y = b^x \to (2)$

Solving (1) and (2), $a^x = b^x, \Rightarrow x = 0 \quad (\because a \neq b)$

When $x = 0; y = a^0 = 1$

The point of intersections is (0, 1)

Differentiate (1) with respect to '*x*', $\frac{dy}{dx} = a^x \log a$

$$\left(\frac{dy}{dx}\right)_{(0,1)} = a^0 \log a = \log a \quad (m1)$$

Differentiate (1) with respect to '*x*', $\frac{dy}{dx} = b^x \log b$

$$\left(\frac{dy}{dx}\right)_{(0,1)} = b^0 \log b = \log b \quad (m2)$$

Let θ be angle between (1) and (2)

$$\tan\theta = \left|\frac{m1 - m2}{1 + m1m2}\right| = \left|\frac{\log a - \log b}{1 + \log a.\log b}\right|$$

$$\theta = \tan^{-1}\left(\left|\frac{\log a - \log b}{1 + \log a.\log b}\right|\right)$$

49. If $u = \log(e^x + e^y)$, find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$

Solution:

$u = \log(e^x + e^y)$ -------- (1)

Differentiate (1) part with respect to *x*,

$$\frac{\partial u}{\partial x} = \frac{1}{e^x + e^y}(e^x) = \frac{e^x}{e^x + e^y}$$

Differentiate (1) part with respect to *y*,

$$\frac{\partial u}{\partial y} = \frac{1}{e^x + e^y}(e^y) = \frac{e^y}{e^x + e^y}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{e^x}{e^x + e^y} + \frac{e^y}{e^x + e^y} = \frac{e^x + e^y}{e^x + e^y} = 1$$

50. Evaluate: $\int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}}$

Solution:

$$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\frac{\sin x}{\cos x}}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx \to (1)$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\pi/3 + \pi/6 - x\right)}}{\sqrt{\sin\left(\pi/3+\pi/6 - x\right)} + \sqrt{\cos\left(\pi/3 + \pi/6 - x\right)}} \quad \left(\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right)$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\pi/2 - x\right)}}{\sqrt{\sin\left(\pi/2 - x\right)} + \sqrt{\cos\left(\pi/2 - x\right)}}\, dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx \to (2)$$

$(1) + (2) \Rightarrow$

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx$$

$$2I = \int_{\pi/6}^{\pi/3} dx = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$\therefore I = \frac{\pi}{12}$$

51. Solve: $\frac{dy}{dx} = 1 + x + y + xy$

Solution:

$$\frac{dy}{dx} = 1 + x + y + xy = (1+x) + y(1+x) = (1+x)(1+y)$$

$$\frac{dy}{(1+y)} = (1+x)dx; \quad \text{integrate both side}$$

$$\int \frac{dy}{(1+y)} = \int (1+x)dx$$

$$log(1+y) = x + \frac{x^2}{2} + c$$

52. Construct the truth table for $(p \wedge q) \vee [\neg(p \wedge q)]$

Solution:

P	q	$p \wedge q$	$\neg(p \wedge q)$	$(p \wedge q) \vee [\neg(p \wedge q)]$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

53. Find the mean and variance of distribution $f(x) = \begin{cases} 3e^{-3x}, & 0 < x < \infty \\ 0, & else\ where \end{cases}$

Solution:

$$Mean = E(x) = \int_{-\infty}^{\infty} x\, f(x)dx$$

$$= \int_{0}^{\infty} x\left(3e^{-3x}\right)dx + 0 = 3\int_{0}^{\infty} x\, e^{-3x} dx$$

$$= 3\left[\frac{1!}{3^{1+1}}\right] \left(\because \int_{0}^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}\right)$$

$$= 3\left(\frac{1}{9}\right) = \frac{1}{3}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$$

$$= \int_{0}^{\infty} x^2\left(3e^{-3x}\right)dx = 3\int_{0}^{\infty} x^2 e^{-3x} dx$$

$$= 3\left(\frac{2!}{3^{2+1}}\right) = 3\left(\frac{2}{3^3}\right) = \frac{2}{9}$$

$$Variance = E(x^2) - [E(x)]^2 = \frac{2}{9} - \left(\frac{1}{3}\right)^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

54. Marks in an aptitude test given to 800 students of a school was found to be normally distributed. 10% of the students scored below 40 marks and 10% of the students scored above 90 marks. Find the number of students who scored between 40 and 90.

Solution:

Let μ be the Mean, σ be the Standard Deviation.

Given 10% of the students below 40,

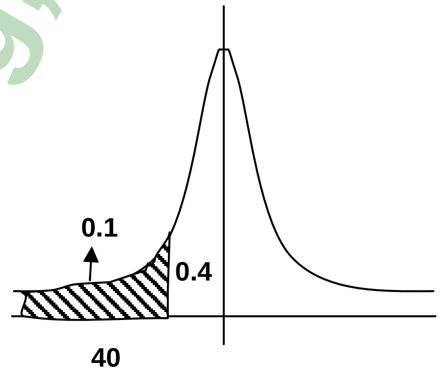
$P(Z < Z_1) = 0.1 \qquad When\ Z_1 = \frac{40-\mu}{\sigma}$

$P(0 < Z < Z_1) = 0.4$

$Z_1 = -1.28$

$\frac{40-\mu}{\sigma} = -1.28$

$40 - \mu = -1.28\sigma \rightarrow (1)$



Also given 10% of the students got more than 90%

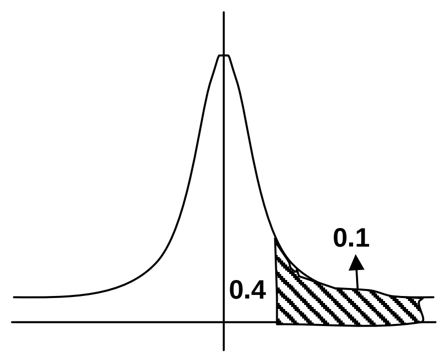
$P(Z_2 < Z) = 0.1 \qquad When\ Z_2 = \frac{90-\mu}{\sigma}$

$\Rightarrow P(0 < Z < Z_2) = 0.4$

$\Rightarrow Z_2 = 1.28$

$Here \quad \frac{90-\mu}{\sigma} = 1.28$

$90 - \mu = 1.28\,\sigma \rightarrow (2)$



$90 - \mu = 1.28\,\sigma \rightarrow (2)$

$40 - \mu = 1.28\,\sigma \rightarrow (1)$

$50 = 2.56\,\sigma$

$\sigma = \frac{50}{2.56}$

$\Rightarrow \quad \sigma = 19.53$

$90 - \mu = (1.28)(19.53)$

$\Rightarrow \quad \mu = 65$

$To\ find\ P(40 < X < 90)$

$$x_1 = 40 \Rightarrow Z_1 = \frac{40-65}{19.53} = -1.28$$

$$x_2 = 90 \Rightarrow Z_2 = \frac{90-65}{19.53} = 1.28$$

$$P(-1.28 < Z < 1.28) = 2\ P(0 < Z < 1.28)$$

$$= 2(0.3997) = 0.7994\ \&$$

$$Number\ of\ students = (0.7994)\times 800 = 639.52 = 640$$

55. a) Find the equation of the hyperbola whose centre is (1, 2). The distance between the directrices is $\frac{20}{3}$, the distance between the foci is 30 and the transverse axis is parallel to y-axis.

Solution:

The transverse axis parallel to y-axis

Centre (h, k) = (1, 2)

Distance between the directions = $\frac{20}{3}$

$$\frac{2a}{e} = \frac{20}{3}$$

$$\frac{a}{e} = \frac{10}{3} \rightarrow (1)$$

Distance between foci is 30

$$2ae = 30$$

$$ae = \frac{30}{2} = 15 \rightarrow (2)$$

From ***(1) & (2)***

$$\frac{a}{e}\times ae = \frac{10}{3}\times 15 \quad \Rightarrow \quad a^2 = 50$$

$$\frac{(1)}{(2)} \Rightarrow \frac{ae}{a/e} = \frac{15}{10/3} \Rightarrow ae\times\frac{e}{a} = 15\times\frac{3}{10}$$

$$\therefore \quad e^2 = \frac{9}{2}; \quad e = \frac{3}{\sqrt{2}}$$

$$b^2 = a^2(e^2 - 1) = 50\left(\frac{9}{2} - 1\right) = 50\times\frac{7}{2} = 175$$

The equation of hyperbola is,

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$
$$\frac{(y-2)^2}{50}-\frac{(x-1)^2}{175}=1$$

55. b) Show that the set of four matrices $\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}-1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix} \& \begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ form an abelian group, under multiplication of matrices.

Solution:

Let $I=\begin{pmatrix}1&0\\0&1\end{pmatrix}, A=\begin{pmatrix}-1&0\\0&1\end{pmatrix}, B=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, c=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$

G = {I, A, B, C}

.	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

$$A^2=AA=\begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}-1&0\\0&1\end{pmatrix}=\begin{pmatrix}1+0&0+0\\0+0&0+1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}=I$$
$$AB=\begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\begin{pmatrix}-1+0&0+0\\0+0&0-1\end{pmatrix}=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}=C$$
$$BB=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\begin{pmatrix}1+0&0+0\\0+0&0+1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}=I$$

i) Closure axiom: All the entries in the table are elements of G. Closure axiom is true.
ii) Associative axiom: Matrix multiplication is always associative.
iii) Identity axiom: The identity element is $I \in G$.
iv) Inverse axiom: The inverse of I, A, B, C are I, A, B, C respectively.
v) Commutative axiom: From the table it satisfies the commutative axiom.

G is an abelian group under matrix multiplication.

PART – C

Note: i) Answer **any ten** questions.

ii) Question No.**70** is **Compulsory** and choose **any nine** questions from the remaining.

10 x 10 = 100

56. Solve the following non-homogeneous system of linear equations by determinant method.

$$2x + y - z = 4; \; x + y - 2z = 0; \; 3x + 2y - 3z = 4$$

Solution:

$$2x + y - z = 4 \rightarrow (1) \quad x + y - 2z = 0 \rightarrow (2) \quad 3x + 2y - 3z = 4 \rightarrow (3)$$

$$\Delta = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ 3 & 2 & -3 \end{vmatrix} = 2(-3+4) - 1(-3+6) - 1(2-3) = 2 - 3 + 1 = 0$$

$$\Delta_x = \begin{vmatrix} 4 & 1 & -1 \\ 0 & 1 & -2 \\ 4 & 2 & -3 \end{vmatrix} = 4(-3+4) - 1(0+8) - 1(0-4) = 4 - 8 + 4 = 0$$

$$\Delta_y = \begin{vmatrix} 2 & 4 & -1 \\ 1 & 0 & -2 \\ 3 & 4 & -3 \end{vmatrix} = 2(0+8) - 4(-3+6) - 1(4-0) = 16 - 12 - 4 = 0$$

$$\Delta_z = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 1 & 0 \\ 4 & 2 & 4 \end{vmatrix} = 2(4+0) - 1(4-0) + 4(2-3) = 8 - 4 - 4 = 0$$

Since $\Delta = 0, \quad and \quad \Delta_x = \Delta_y = \Delta_z = 0$ but one of 2 x 2 minor is $\Delta \neq 0, \left(\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2 - 1 = 1 \neq 0 \right)$. The system is consistent and has infinitely many solutions.

The system reduces to two equations. Assume that $z = k$,

$$(1) \Rightarrow \quad 2x + y - k = 4 \Rightarrow 2x + y = 4 + k$$

$$(2) \Rightarrow \quad x + y - 2k = 0 \Rightarrow x + y = 2k$$

$$\Delta = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2 - 1 = 1 \neq 0; \quad \Delta_x = \begin{vmatrix} 4+k & 1 \\ 2k & 1 \end{vmatrix} = 4 + k - 2k = 4 - k; \quad \Delta_y = \begin{vmatrix} 2 & 4+k \\ 1 & 2k \end{vmatrix} = 4k - 4 - k = 3k - 4$$

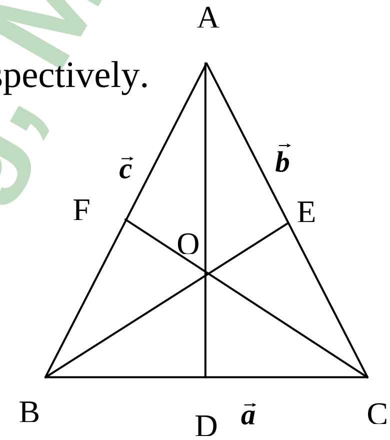
$$x = \frac{\Delta_x}{\Delta} = \frac{4-k}{1} = 4 - k; \quad y = \frac{\Delta_y}{\Delta} = \frac{3k-4}{1} = 3k - 4$$

The solution is $(x, y, z) = (4 - k, 3k - 4, k), k \in R$.

57. Altitudes of a triangle are concurrent. Prove by vector method.

Solution:

Let ABC be a triangle and AD, BE, be its altitudes. It is enough to prove that CO is perpendicular to AB.



Let the position vectors of A, B, C be $\vec{a},\vec{b},\vec{c}$ respectively.

$\therefore \overrightarrow{OA}=\vec{a}, \quad \overrightarrow{OB}=\vec{b}, \quad \overrightarrow{OC}=\vec{c},$

$Now \quad AD \perp BC$

$\Rightarrow \overrightarrow{OA} \perp \overrightarrow{BC}$

$\Rightarrow \vec{a} \bullet \left(\vec{c}-\vec{b}\right)=0$

$\Rightarrow \vec{a} \bullet \vec{c}-\vec{a} \bullet \vec{b}=0 \quad \rightarrow (1)$

$Now \quad BE \perp CA \Rightarrow \overrightarrow{OB} \perp \overrightarrow{CA}$

$\Rightarrow \overrightarrow{OB} \bullet \overrightarrow{CA}$

$\Rightarrow \vec{b} \bullet \left(\vec{a}-\vec{c}\right)=0$

$\Rightarrow \vec{b} \bullet \vec{a}-\vec{b} \bullet \vec{c}=0 \quad \rightarrow (2)$

Adding ***(1) & (2)*** , we get

$\vec{a} \bullet \vec{c}-\vec{b} \bullet \vec{c}=0 \quad \Rightarrow \left(\vec{a}-\vec{b}\right)\bullet \vec{c}=0$

$\overrightarrow{BA} \bullet \overrightarrow{OC}=0 \quad \Rightarrow \quad \overrightarrow{OC} \perp \overrightarrow{AB}$

Hence the three altitudes are concurrent.

58. Find all the values $\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{3/4}$. Hence prove that the product of the four values is 1.

Solution:

Let $\frac{1}{2}+i\frac{\sqrt{3}}{2}=r(\cos\theta+i\sin\theta)$

Equating the real & imaginary part,

$r\cos\theta=\frac{1}{2}; r\sin\theta=\frac{\sqrt{3}}{2}$

Squaring and adding the above, $r^2=\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2=\frac{1}{4}+\frac{3}{4}=1$

$r=1$

$\cos\theta=1/2; \sin\theta=\sqrt{3}/2$; θ lies in I quadrant $\theta=\frac{\pi}{3}$

$$\frac{1}{2}+i\frac{\sqrt{3}}{2}=1\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

$$\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{3/4}=\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{3/4}$$

$$=(\cos\pi+i\sin\pi)^{1/4}$$

$$=(\cos(2k\pi+\pi)+i\sin(2k\pi+\pi))^{1/4},\quad k=0,1,2,3$$

$$=(\cos(2k+1)\pi+i\sin(2k+1)\pi)^{1/4},\quad k=0,1,2,3$$

$$=\left(\cos\frac{(2k+1)}{4}\pi+i\sin\frac{(2k+1)}{4}\pi\right),\quad k=0,1,2,3$$

The roots are,

$$\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right),\ \left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right),\ \left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right),\ \left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right).$$

Product of roots= $\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$

$$=\cos\left(\frac{\pi}{4}+\frac{3\pi}{4}+\frac{5\pi}{4}+\frac{7\pi}{4}\right)+i\sin\left(\frac{\pi}{4}+\frac{3\pi}{4}+\frac{5\pi}{4}+\frac{7\pi}{4}\right)$$

$$=\cos\left(\frac{16\pi}{4}\right)+i\sin\left(\frac{16\pi}{4}\right)=\cos 4\pi+i\sin 4\pi=1+i(0)=1$$

59. Find the axis, vertex, focus, directrix, equation of the latus rectum, length of the latus rectum of the parabola $x^2+8y-2x+17=0$ and draw the diagram.

Solution:

The equation of parabola is, $x^2+8y-2x+17=0$

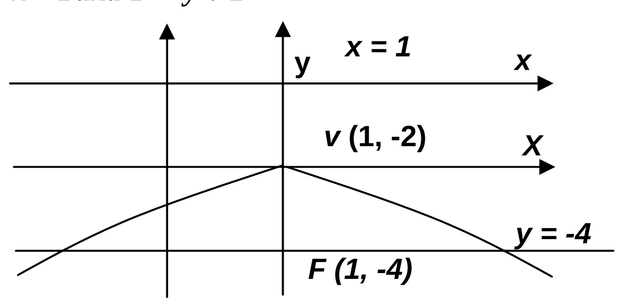
$$x^2-2x=-8y-17$$

$$(x-1)^2=-8y-17+1=-8y-16$$

$$(x-1)^2=-8(y+2)$$

$$X^2=-8Y,\quad \text{where } X=x-1 \text{ and } Y=y+2$$

$$4a=8\Rightarrow a=2$$



	Referred to X and Y axes	Referred to x and y axes x = X+1, y = Y-2
Axis	*X = 0*	*x = 1*
Vertex	*(0, 0)*	*V(1, -2)*
Focus	*(0, -a) = (0, -2)*	*F(1, -4)*
Directices	$Y = a \Rightarrow Y = 2$	*y = 0*
Equation of latus rectum	$Y = a \Rightarrow Y = -2$	*y = -4*

Length of latus rectum = 4a = 4(2) = 8

60. A kho - kho player in a practice session while running realizes that the sum of the distances from the two kho - kho poles from him is always 8m. Find the equation of the path traced by him if the distance between the poles is 6m.

Solution:

From the given data, the kho – kho poles be taken as the points F_1 and F_2.

Let $P(x_1, y_1)$ be the position of player at any instant.

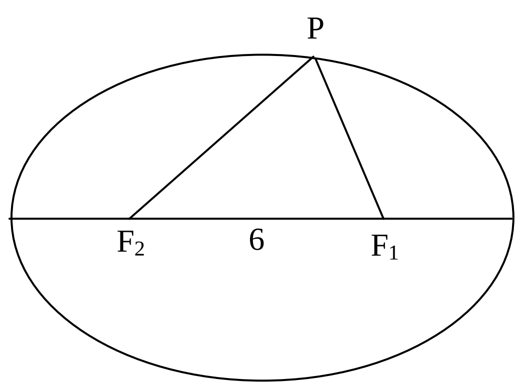
$$F_1P + F_2P = 2a$$

$$8 = 2a \Rightarrow a = 4$$

$$F_1F_2 = 2ae; \quad 6 = 2ae$$

$$ae = 3; \text{ and } e = \tfrac{3}{4} \quad \because a = 4$$

$$b^2 = a^2(1 - e^2) = 16\left(1 - \frac{9}{16}\right) = 16 \times \frac{7}{16} = 7$$



The equation of path is $\frac{x^2}{16} + \frac{y^2}{7} = 1$.

61. Find the equation of the hyperbola if its asymptotes are parallel to $x + 2y - 12 = 0$ and $x - 2y + 8 = 0$, (2, 4) is the centre of the hyperbola and it passes through (2, 0).

Solution:

The equation parallel to the line $x + 2y - 12 = 0$ is $x + 2y = k$

The centre ***(2, 4)*** lies on it. $2 + 8 = k \Rightarrow k = 10.$

$\therefore$ One of the asymptote is $x + 2y - 10 = 0$

Similarly the equation parallel to the line

$$x - 2y + 8 = 0 \text{ is } x - 2y = l$$

The Centre (2, 4) lies on it, $2 - 8 = l \Rightarrow l = -6$

$\therefore$ another asymptote is $x - 2y + 6 = 0$

$\therefore$ The combined equation of the asymptote is

$$(x + 2y - 10)(x - 2y + 6) = 0$$

The equation of the hyperbola is,

$$(x + 2y - 10)(x - 2y + 6) = a$$

It passes through the point ***(2, 0)***

(2+0-10) (2-0+6) = a

(-8)(8) = a; a = -64

$\therefore$The equation of the hyperbola is,

(x+2y-10) (x-2y+6) = -64

$$x^2 + 2xy - 10x - 2xy - 4y^2 + 20y + 6x + 12y - 60 = -64$$

$$x^2 - 4y^2 - 4x + 32y + 4 = 0$$

62. Find the local minimum and maximum values of $f(x) = x^4 - 3x^3 + 3x^2 - x$

Solution:

$$f(x) = x^4 - 3x^3 + 3x^2 - x$$

$$f'(x) = 4x^3 - 9x^2 + 6x - 1$$

$$f'(x) = 0 \Rightarrow 4x^3 - 9x^2 + 6x - 1 = 0$$

$$(x - 1)(x - 1)(4x - 1) = 0$$

$$x = 1, \quad 1, \quad 1/4$$

$$\begin{array}{r|rrrr} 1 & 4 & -9 & 6 & -1 \\ & 0 & 4 & -5 & 1 \\ \hline 1 & 4 & -5 & 1 & 0 \\ & 0 & 4 & -1 & \\ \hline & 4 & -1 & 0 & \end{array}$$

The turning points are $1, \quad 1, \quad 1/4$

when $x = 1, \quad f(1) = 1 - 3 + 3 - 1 = 0$

when $x = 1/4, \quad f(1/4) = (1/4)^4 - 3(1/4)^3 - 3(1/4)^2 - \frac{1}{4}$

$$= \frac{1}{256} - \frac{3}{64} + \frac{3}{16} - \frac{1}{4}$$

$$= \frac{1 - 12 + 48 - 64}{256} = \frac{-27}{256}$$

The stationary points are $(1, \quad 0)$, $\left(\frac{1}{4}, \quad \frac{-27}{256}\right)$

$$f''(x) = 12x^2 - 18x + 6$$

$$f''(1) = 12(1)^2 - 18(1) + 6 = 0$$

Second derivative gives no information about the extremum nature of at $x = 1$

when $x = \frac{1}{4}, \quad f''\left(\frac{1}{4}\right) = 12\left(\frac{1}{4}\right)^2 - 18\left(\frac{1}{4}\right) + 6$

$$= \frac{3}{4} - \frac{9}{2} + 6 = \frac{3 - 18 + 24}{4} = \frac{9}{4} > 0$$

$$\therefore \left(\frac{1}{4}, \quad \frac{-27}{256}\right) \text{ is a minimum point.}$$

63. Use differentials to field an appropriate value for the given number $y = \sqrt[3]{1.02} + \sqrt[4]{1.02}$

Solution:

$$y = \sqrt[3]{1.02} + \sqrt[4]{1.02}$$

$$y = u + v, \quad where \quad u = \sqrt[3]{1.02}, \quad v = \sqrt[4]{1.02}$$

$$u = \sqrt[3]{1.02}$$

$$Let \quad u = f(x) = x^{\frac{1}{3}}$$

$$du = \frac{1}{3} x^{\frac{1}{3} - 1} \quad dx = \frac{1}{3} x^{-\frac{2}{3}} \quad dx$$

Since $f(1) = 1$, Take $x = 1, \quad dx = \Delta x = 0.02$

$$du = \frac{1}{3}(1)^{-\frac{2}{3}}(0.02) = \frac{0.02}{3} = 0.00667$$

$$\sqrt[3]{1.02} = f(1.02) = f(1) + du = 1 + 0.00667 = 1.00667$$

$$v = \sqrt[4]{1.02}$$

$$Let \quad v = f(x) = x^{\frac{1}{4}}$$

$$dv = \frac{1}{4} x^{\frac{1}{4} - 1} \quad dx = \frac{1}{4} x^{-\frac{3}{4}} \quad dx$$

Since $f(1) = 1$, Take $x = 1, \quad dx = \Delta x = 0.02$

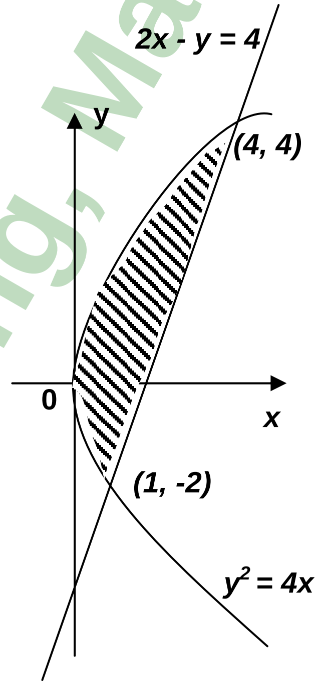
$$dv = \frac{1}{4}(1)^{-\frac{3}{3}}(0.02) = \frac{0.02}{4} = 0.005$$

$$\sqrt[4]{1.02} = f(1.02) = f(1) + dv = 1 + 0.005 = 1.005 \text{ S}$$

$$y = u + v = 1.00667 + 1.005 = 2.01167 \quad (app.)$$

64. Find the area of the region bounded by the parabola $y^2 = 4x$ and the line $2x - y = 4$.

Solution:



The given curves are

$y^2 = 4x \rightarrow (1);\quad 2x - y = 4 \rightarrow (2)$

$(2) \Rightarrow \quad y = 2x - 4$

substitute value of y in ***(1)***

$(2x-4)^2 = 4x$

$16 + 4x^2 - 16x - 4x = 0$

$4x^2 - 20x + 16 = 0$

$x^2 - 5x + 4 = 0$

$(x-1)\ (x-4) = 0$

$x = 1, 4$

when $x = 1, \quad y = -2$

$x = 4, \quad y = 4$

The point of intersection are $(1, -2), \quad (4, 4)$

The required area = $\int_c^d (x_1 - x_2)\ dy$

$$= \int_{-2}^{4}\left(\frac{y+4}{2} - \frac{y^2}{4}\right) dy = \frac{1}{4}\int_{-2}^{4}(2y + 8 - y^2)\ dy = \frac{1}{4}\left[\frac{2y^2}{2} + 8y - \frac{y^3}{3}\right]_{-2}^{4}$$

$$= \frac{1}{4}\left[\left(16 + 32 - \frac{64}{3}\right) - \left(4 - 16 + \frac{8}{3}\right)\right]$$

$$= \frac{1}{4}\left[48 - \frac{64}{3} + 12 - \frac{8}{3}\right] = \frac{1}{4}\left[60 - \frac{72}{3}\right] = \frac{1}{4}[60 - 24] = \frac{36}{4} = 9 \quad \textit{square units}$$

65. Find the surface area of the solid generated by revolving the cycloid $x = a(t + \sin t)$, $y = a(1 + \cos t)$ about its base *(x – axis)*.

Solution:

The given curves are $x = a(t + \sin t)$ and $y = a(1 + \cos t)$

$y = 0 \quad \Rightarrow \quad a(1 + \cos t) = 0$

$1 + \cos t = 0 \Rightarrow \quad \cos t = -1 \quad \Rightarrow t = -\pi, \quad \pi$

$$\frac{dx}{dt} = a(1+\cos t) \quad \frac{dy}{dt} = a(-\sin t) = -a\sin t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{a^2(1+\cos t)^2 + a^2\sin^2 t}$$

$$= \sqrt{a^2(1+\cos^2 t + 2\cos t + \sin^2 t)}$$

$$= \sqrt{a^2(2+2\cos t)} = \sqrt{2a^2(1+\cos t)}$$

$$= \sqrt{2a^2\left(2\cos^2 t/2\right)} = 2a\cos t/2$$

The required surface area $= \int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$$= \int_{-\pi}^{\pi} 2\pi\, a(1+\cos t) \quad 2a \quad \cos\frac{t}{2} \quad dt$$

$$= \int_{-\pi}^{\pi} 2\pi\, a\left(2\cos^2\frac{t}{2}\right) \quad 2a \quad \cos\frac{t}{2} \quad dt$$

$$= 8\pi\, a^2 \int_{-\pi}^{\pi} \cos^3\frac{t}{2} \quad dt$$

$$= 16\pi\, a^2 \int_{0}^{\pi/2} \cos^3\frac{t}{2} \quad dt \qquad put \quad u = \frac{t}{2}, \quad du = \frac{dt}{2}, \quad dt = 2du, \quad t = 0, \quad u = 0, \quad t = \pi, \quad u = \frac{\pi}{2}$$

$$= 32\pi\, a^2 \int_{0}^{\pi/2} \cos^3 u \quad du = 32\pi\, a^2 * \frac{2}{3} = \frac{64\pi\, a^2}{3} \quad square\ units.$$

66. Solve: $(x^2 + y^2)dx + 3xy\, dy = 0$

Solution: $(x^2 + y^2)dx + 3xy\, dy = 0$

$$3xy\, dy = -(x^2 + y^2)\, dx$$

$$Put \quad y = vx \quad then\ \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\frac{dy}{dx} = -\frac{(x^2 + y^2)}{3xy}$$

$$v + x\frac{dv}{dx} = -\frac{(x^2 + v^2x^2)}{3x(vx)} = -\frac{x^2(1+v^2)}{3x^2v} = -\frac{(1+v^2)}{3v}$$

$$x\frac{dv}{dx} = -\frac{(1+v^2)}{3v} - v = \frac{-1- \quad v^2 - 3v^2}{3v} = \frac{-1-4v^2}{3v} = -\frac{(1+4v^2)}{3v}$$

$$\int \frac{3v}{4v^2+1}dv = -\int \frac{dx}{x}$$

$$3*\frac{1}{8}\int \frac{8v}{4v^2+1}dv = -\int \frac{dx}{x}$$

$$\frac{3}{8}\log(4v^2+1) = -\log x + \log c$$

$$\frac{3}{8}\log(4v^2+1) + \log x = \log c$$

$$\log(4v^2+1)^{\frac{3}{8}}\ x = \log c$$

$$(4v^2+1)^{\frac{3}{8}}\ x = c$$

$$\text{substitute} \quad v = \frac{y}{x},$$

$$\left(4\frac{y^2}{x^2}+1\right)^{\frac{3}{8}} x = c$$

$$\left(\frac{4y^2+x^2}{x^2}\right)^{\frac{3}{8}} x = c$$

Raising to power 8, $(x^2+4y^2)^3\ x^2 = c$

67. The sum of ₹.1000 is compounded continuously, the nominal rate of interest being four percent per annum. In how many years will the amount be twice the original principal? $(\log_e 2 = 0.6931)$

Solution:

Rate of interest = 4%

$$\Rightarrow \frac{dx}{dy} = 0.04\,x \Rightarrow \frac{dx}{x} = 0.04\,dt$$

Integrating, $\log x = 0.04t + c$

$$\Rightarrow \quad x = c\,e^{0.04t} \quad \rightarrow (1)$$

$$when,\ t = 0,\ x = x_o$$

$$\Rightarrow c\,e^{o} = x_o \quad \Rightarrow \quad c = x_o$$

$$when \quad x_o = 2x_o$$

$$(1) \Rightarrow \quad 2x_o = x_o\,e^{0.04t} \quad \Rightarrow \quad 2 = e^{0.04t}$$

Taking log on both sides,

$$\log_e 2 = 0.04\,t$$
$$0.04\,t = 0.6931$$
$$t = \frac{0.6931}{0.04} = 17.3275 = 17\,years$$

The amount will be twice the original principal in 17 years.

68. Show that the set G of all matrices of the form $\begin{pmatrix} x & x \\ x & x \end{pmatrix}$, where $x \in R - \{0\}$, is a group under matrix multiplication.

Solution:

Let, $G = \left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} \Big/ x \in R - \{0\} \right\}$

i) Closure axiom: Let ***A, B*** $\in$ ***G***

$$A = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \quad B = \begin{pmatrix} y & y \\ y & y \end{pmatrix} \quad x, y \in R - \{0\}$$

$$AB = \begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} y & y \\ y & y \end{pmatrix} = \begin{pmatrix} xy + xy & xy + xy \\ xy + xy & xy + xy \end{pmatrix}$$

$$= \begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} \in G \quad (\because x \neq 0, y \neq 0 \quad \Rightarrow \quad 2xy \neq 0)$$

Closure axiom is true.

ii) Associative axiom:

Matrix multiplication is always associative.

iii) Identity axiom:

$Let, E = \begin{pmatrix} e & e \\ e & e \end{pmatrix} \in G$ be the identity of ***G.***

By definition, $AE = EA = A \Rightarrow AE = A$

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix} \begin{pmatrix} e & e \\ e & e \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$\begin{pmatrix} 2xe & 2xe \\ 2xe & 2xe \end{pmatrix} = \begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

$$2xe = x \quad \Rightarrow e = 1/2 \quad (\because x \neq 0)$$

The identity element is $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \in G$

iv) Inverse axiom:

$$Let,\ A^{-1} = \begin{pmatrix} y & y \\ y & y \end{pmatrix} \in G$$ be inverse of ***A***

By definition, $AA^{-1} = A^{-1}A = E; \quad AA^{-1} = E$

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix}\begin{pmatrix} y & y \\ y & y \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\begin{pmatrix} 2xy & 2xy \\ 2xy & 2xy \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$2xy = 1/2; \quad y = \frac{1}{4}x$$

$$A^{-1} = \begin{pmatrix} 1/4\,x & 1/4\,x \\ 1/4\,x & 1/4\,x \end{pmatrix} \in G$$

there fore, G is a group under matrix multiplication.

69. A random variable X has the following probability mass function

x	**0**	**1**	**2**	**3**	**4**	**5**	**6**
P(X=x)	**k**	**3k**	**5k**	**7k**	**9k**	**11k**	**13k**

a) Find k
b) Evaluate $P(X < 4)$, $P(X \geq 5)$ and $P(3 < X \leq 6)$
c) What is the smallest value of x for which $P(X \leq x) > ½$?

Solution:

a) Since ***P(X = x)*** is a probability mass function,

$$\sum_{x=0}^{6} P(X = x) = 1$$

$$P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6) = 1$$

$$k + 3k + 5k + 7k + 9k + 11k + 13k = 1$$

$$49k = 1; \quad k = 1/49$$

b) $$P(X < 4) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

$$= \frac{1}{49} + \frac{3}{49} + \frac{5}{49} + \frac{4}{49} = \frac{16}{49}$$

$$P(X \geq 5) = P(X = 5) + P(X = 6) = \frac{11}{49} + \frac{13}{49} = \frac{24}{49}$$

$$P(3 < X \leq 6) = P(X = 4) + P(X = 5) + P(X = 6)$$

$$= \frac{9}{49} + \frac{11}{49} + \frac{13}{49} = \frac{33}{49}$$

c) The minimum value of ***x*** may be determined by trial and error method.

$$P(X \le 0) = \frac{1}{49} < \frac{1}{2}$$

$$P(X \le 1) = \frac{4}{49} < \frac{1}{2}$$

$$P(X \le 2) = \frac{9}{49} < \frac{1}{2}$$

$$P(X \le 3) = \frac{16}{49} < \frac{1}{2}$$

$$P(X \le 4) = \frac{25}{49} > \frac{1}{2}$$

There fore, the smallest value of x for which $P(X \le x) > \frac{1}{2}$ is 4.

70. a) Derive the equation of the plane in the intercept form.

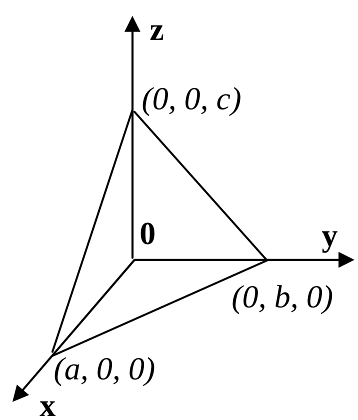
Solution:

Let a, b, c be the x, y and z intercept of the plane respectively.

The plane passing through the point ***(a, 0, 0)***, ***(0, b, 0)*** and ***(0, 0, c).***

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

Vector form:



The equation of the plane

$$\vec{r} = (1-s-t)\vec{a} + s\,\vec{b} + t\,\vec{c}$$

$$\vec{r} = (1-s-t)\,a\,\vec{i} + s\,b\,\vec{j} + t\,c\,\vec{k}$$

$$x\vec{i} + y\vec{j} + z\vec{k} = (1-s-t)\,a\,\vec{i} + s\,b\,\vec{j} + t\,c\,\vec{k}$$

$$x = (1-s-t)\,a; \quad y = s\,b; \quad z = t\,c$$

$$\frac{x}{a} = 1-s-t; \quad \frac{y}{b} = s; \quad \frac{z}{c} = t$$

$$\frac{x}{a} + s + t = 1; \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Cartesian form:

The equation of the plane

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-a & y-0 & z-0 \\ 0-a & b-0 & 0-0 \\ 0-a & 0-0 & c-0 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-a & y & z \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = 0$$

$$(x-a)(bc-0) - y(ac-0) + z(0+ab) = 0$$

$$bcx - abc + acy + abz = 0$$

$$bcx + acy + abz = abc$$

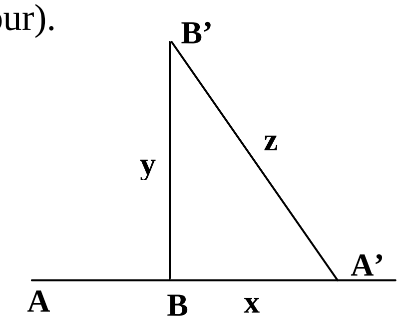
$$\div abc, \text{ then } \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

70. b) At noon, ship A is 100 km west of ship B. Ship A is sailing east at 35 km/hr and ship B is sailing north at 25 km / hr. How fast is the distance between the ships changing at 4.00 pm?

Solution:

Let A, B be position of ship A and ship B at noon. A' B' be the new position of ship A and B.

Let x be the distance between B and A', y be the distance between B and B', z be the distance between A' and B' at time t (in hour).



Given $\frac{dx}{dt} = 35\ km/hr$, $\frac{dy}{dt} = 25\ km/hr$

From the diagram, $z^2 = x^2 + y^2 \longrightarrow (1)$

To find $\frac{dz}{dt}$, when $t = 4.00\ pm$

Differentiating *(1)* with respect to 't', we get

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} \quad \longrightarrow (2)$$

when $t = 4, \quad x = 4 \times 35 - 100 = 40, \quad y = 4 \times 25 = 100$

$$(1) \Rightarrow z^2 = 40^2 + 100^2 = 1600 + 10000 = 11600$$

$$when \quad z = 20\sqrt{29}$$

$$(2) \Rightarrow 20\sqrt{29}\frac{dz}{dt} = (40)(35) + 100(25)$$

$$= 1400 + 2500 = 3900$$

$$\frac{dz}{dt} = \frac{3900}{20\sqrt{29}} = \frac{195}{\sqrt{29}}$$

there fore, the distance between the ship is changing at the rate of $\frac{195}{\sqrt{29}}\ km/hr$.